AN ℓ_2-BASED PROOF OF GABORIAU'S THEOREM

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ABSTRACT. We present the ℓ_2-based proof of the "Treeings Realize Cost" theorem due to Gaboriau.

This text is an outcome of the Operator Algebras Learning Seminar held at the UIUC mathematics department in the Spring of 2018. It is mostly based on Section 8 of Gaboriaus's notes [Gab16]. Some arguments pertaining to Hilbert modules and von Neumann dimension are inspired by the paper [Eck00] of Eckmann.

Throughout, E is a countable Borel equivalence relation on a standard Borel space X and μ is an E-invariant probability Borel measure on X. We write $[x]$ for the E-class of $x \in X$ and $x \sim y$ to indicate that x and y are E-equivalent. The set of all E-classes is denoted by X/E. The following is a fundamental result of Gaboriau:

Theorem 0.1 (Gaboriau [Gab98])**.** *Suppose that T is a Borel treeing of E. Then $C_\mu(E) = C_\mu(T)$.*

Gaboriau's original proof of Theorem 0.1 was combinatorial and used the technique of *foldings*. Later, Gaboriau found another proof of Theorem 0.1 using ℓ_2-methods. The purpose of this note is to give a self-contained presentation of that new proof. A standard argument that we do not repeat here reduces Theorem 0.1 to the following statement:

Theorem 0.2. *Suppose that T and G are graphings of E such that:*
- *$\Delta(T)$ and $\Delta(G)$ are finite;*
- *there is some $c > 0$ such that for all $x, y \in X$ with $x \sim y$, we have*
$$c^{-1} \cdot \mathrm{dist}_T(x, y) \leqslant \mathrm{dist}_G(x, y) \leqslant c \cdot \mathrm{dist}_T(x, y);$$
- *T is acyclic (i.e, it is a treeing).*

Then $C_\mu(T) \leqslant C_\mu(G)$.

In the sequel, we prove Theorem 0.2.

1. HILBERT E-MODULES

For a countable set S, we let $\ell_2 S$ denote, as usual, the Hilbert space of all maps $f \colon S \to \mathbb{R}$ satisfying $\sum_{s \in S} |f(s)|^2 < \infty$, equipped with the inner product $\langle f, g \rangle := \sum_{s \in S} f(s)g(s)$. (It is possible to work over $\mathbb{C}$ instead, but that will not be necessary for our purposes.) For $n \in \mathbb{N}^+$, we let
$$(\ell_2 S)^n := \underbrace{\ell_2 S \oplus \cdots \oplus \ell_2 S}_{n \text{ times}}.$$

Definition 1.1. A **Hilbert E-module** M is an assignment $\mathcal{O} \mapsto M_\mathcal{O}$ to each $\mathcal{O} \in X/E$ of a closed subspace $M_\mathcal{O} \subseteq (\ell_2 \mathcal{O})^n$ (where $n \in \mathbb{N}^+$ is fixed) that is Borel in the following sense: Let $\Gamma \curvearrowright X$ be a Borel countable group action that generates E. Then the set

$$\{(x, f_1, \ldots, f_n) \in X \times (\mathbb{R}^\Gamma)^n : \text{the tuple of maps } ([x] \to \mathbb{R} \colon \gamma \cdot x \mapsto f_i(\gamma))_{i=1}^n \text{ is in } M_{[x]}\}$$

is Borel.

Exercise 1.2. Show that the above definition is independent of the choice of the action $\Gamma \curvearrowright X$.

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We write (somewhat ambiguously) $(\ell_2 E)^n$ to denote the Hilbert E-module that assigns to each E-class $\mathcal{O}$ the entire space $(\ell_2 \mathcal{O})^n$, and set $\ell_2 E := (\ell_2 E)^1$. Given Hilbert E-modules M and N, we write $M \subseteq N$ and say that M is a **submodule** of N if $M_\mathcal{O} \subseteq N_\mathcal{O}$ for all $\mathcal{O} \in X/E$.

Definition 1.3. Let $M \subseteq (\ell_2 E)^m$ and $N \subseteq (\ell_2 E)^n$ be Hilbert E-modules. A **fibered map** $\varphi \colon M \to N$ is a Borel assignment $\mathcal{O} \mapsto \varphi_\mathcal{O}$ to each $\mathcal{O} \in X/E$ of a bounded linear operator $\varphi_\mathcal{O} \colon M_\mathcal{O} \to N_\mathcal{O}$ such that the **operator norm** of φ, given by

$$\|\varphi\| := \sup_{\mathcal{O} \in X/E} \|\varphi_\mathcal{O}\|,$$

is finite. A fibered map $\varphi \colon M \to N$ is an **isomorphism** if each $\varphi_\mathcal{O} \colon M_\mathcal{O} \to N_\mathcal{O}$ is an isometry of Hilbert spaces. If there is an isomorphism $\varphi \colon M \to N$, then we say that M and N are **isomorphic** and write $M \cong N$.

Exercise 1.4. Explain what it means for an assignment $\mathcal{O} \mapsto \varphi_\mathcal{O}$ to be Borel.

Remark. For simplicity, we deliberately made our definition of the norm of a fibered map $\varphi \colon M \to N$ more restrictive than necessary: It would suffice to require that $\|\varphi_\mathcal{O}\|$, rather than being uniformly bounded, is "square integrable" in an appropriate sense.

Theorem 1.5. *Let $M \subseteq (\ell_2 E)^m$ and $N \subseteq (\ell_2 E)^n$ be Hilbert E-modules. Suppose that there exists a fibered map $\varphi \colon M \to N$ such that for each $\mathcal{O} \in X/E$, the operator $\varphi_\mathcal{O} \colon M_\mathcal{O} \to N_\mathcal{O}$ is injective and its image $\mathrm{im}(\varphi_\mathcal{O})$ is dense in $N_\mathcal{O}$. Then $M \cong N$.*

Proof. Consider any $\mathcal{O} \in X/E$. The operator $\varphi_\mathcal{O}^* \circ \varphi_\mathcal{O} \colon M_\mathcal{O} \to M_\mathcal{O}$ is positive and its image $\mathrm{im}(\varphi_\mathcal{O}^* \circ \varphi_\mathcal{O})$ is dense in $M_\mathcal{O}$. Therefore, there exists a unique positive self-adjoint operator $\psi_\mathcal{O} \colon M_\mathcal{O} \to M_\mathcal{O}$ with $\psi_\mathcal{O}^2 = \varphi_\mathcal{O}^* \circ \varphi_\mathcal{O}$. This gives a fibered map $\psi \colon M \to M$.

Exercise 1.6. Verify the above claim.

Put $\xi_\mathcal{O} := \varphi_\mathcal{O} \circ \psi_\mathcal{O}^{-1} \colon \mathrm{im}(\psi_\mathcal{O}) \to N_\mathcal{O}$. Then $\mathrm{im}(\xi_\mathcal{O})$ is dense in $N_\mathcal{O}$ and for all $f, g \in \mathrm{im}(\psi_\mathcal{O})$,

$$\begin{aligned}
\langle \xi_\mathcal{O}(f), \xi_\mathcal{O}(g) \rangle &= \langle (\varphi_\mathcal{O} \circ \psi_\mathcal{O}^{-1})(f), (\varphi_\mathcal{O} \circ \psi_\mathcal{O}^{-1})(g) \rangle \\
&= \langle \psi_\mathcal{O}^{-1}(f), (\varphi_\mathcal{O}^* \circ \varphi_\mathcal{O} \circ \psi_\mathcal{O}^{-1})(g) \rangle \\
&= \langle \psi_\mathcal{O}^{-1}(f), \psi_\mathcal{O}(g) \rangle = \langle f, g \rangle,
\end{aligned}$$

where the last equality uses that $\psi_\mathcal{O}$ is self-adjoint. Therefore, $\xi_\mathcal{O}$ is an isometric isomorphism between $\mathrm{im}(\psi_\mathcal{O})$ and $\mathrm{im}(\varphi_\mathcal{O})$. Since $\mathrm{im}(\psi_\mathcal{O})$ is dense in $M_\mathcal{O}$, while $\mathrm{im}(\varphi_\mathcal{O})$ is dense in $N_\mathcal{O}$, we can extend $\xi_\mathcal{O}$ by continuity to an isometric isomorphism $\overline{\xi}_\mathcal{O} \colon M_\mathcal{O} \to N_\mathcal{O}$. This gives the desired isomorphism $\overline{\xi} \colon M \to N$. ∎

2. TRACES

Definition 2.1. Let $\varphi \colon \ell_2 E \to \ell_2 E$ be a fibered map. Define the **trace** $\mathrm{tr}(\varphi)$ of φ by the formula

$$\mathrm{tr}(\varphi) := \int_X \langle \varphi_{[x]}(1_x), 1_x \rangle \, \mathrm{d}\mu(x),$$

where for $x \in X$, we use 1_x to denote the map $[x] \to \mathbb{R}$ sending x to 1 and every $y \in [x] \setminus \{x\}$ to 0.

Exercise 2.2. Show that for all $\varphi \colon \ell_2 E \to \ell_2 E$, we have

$$\mathrm{tr}(\varphi) = \int_X \varphi_{[x]}(1_x)(x) \, \mathrm{d}\mu(x).$$

Exercise 2.3. Show that for all $\varphi \colon \ell_2 E \to \ell_2 E$, we have $|\mathrm{tr}(\varphi)| \leqslant \|\varphi\|$.

Lemma 2.4. *Let $\varphi, \psi \colon \ell_2 E \to \ell_2 E$ be fibered maps. Then*

$$\mathrm{tr}(\varphi \circ \psi) = \mathrm{tr}(\psi \circ \varphi). \tag{2.1}$$

Proof. For $\mathcal{O} \in X/E$ and $x, y \in \mathcal{O}$, define
$$A(x,y) := \psi_{\mathcal{O}}(1_x)(y) \cdot \varphi_{\mathcal{O}}(1_y)(x).$$

Consider any $x \in X$. We have
$$\psi_{[x]}(1_x) = \sum_{y \sim x} \psi_{[x]}(1_x)(y) \cdot 1_y.$$

Therefore, since $\varphi_{[x]}$ is linear, we obtain
$$(\varphi \circ \psi)_{[x]}(1_x) = \varphi_{[x]}(\psi_{[x]}(1_x)) = \varphi_{[x]}\left(\sum_{y \sim x} \psi_{[x]}(1_x)(y) \cdot 1_y\right) = \sum_{y \sim x} \psi_{[x]}(1_x)(y) \cdot \varphi_{[x]}(1_y),$$

which yields
$$(\varphi \circ \psi)_{[x]}(1_x)(x) = \sum_{y \sim x} \psi_{[x]}(1_x)(y) \cdot \varphi_{[x]}(1_y)(x) = \sum_{y \sim x} A(x,y).$$

Note that
$$\sum_{y \sim x} |A(x,y)| \leqslant \|\psi\|\|\varphi\| < \infty.$$

Applying the same reasoning to the composition $\psi \circ \varphi$, we conclude that (2.1) is equivalent to
$$\int_X \sum_{y \sim x} A(x,y)\, \mathrm{d}\mu(x) = \int_X \sum_{y \sim x} A(y,x)\, \mathrm{d}\mu(x). \tag{2.2}$$

Now it is not hard to see that (2.2) follows from the E-invariance of μ. Indeed, let $f_0, f_1, \ldots$ be a sequence of Borel partial involutions $X \rightharpoonup X$ that generates E. We may assume that $f_n(x) \neq f_m(x)$ for all $n \neq m$ and $x \in \mathrm{dom}(f_n) \cap \mathrm{dom}(f_m)$. Set $D_n := \mathrm{dom}(f_n)$. Then
$$\int_X \sum_{y \sim x} A(x,y)\, \mathrm{d}\mu(x) = \int_X \sum_{n \in \mathbb{N}} [x \in D_n] \cdot A(x, f_n(x))\, \mathrm{d}\mu(x)$$
$$= \sum_{n \in \mathbb{N}} \int_{x \in D_n} A(x, f_n(x))\, \mathrm{d}\mu(x)$$
$$= \sum_{n \in \mathbb{N}} \int_{x \in D_n} A(f_n(x), x)\, \mathrm{d}\mu(x)$$
$$= \int_X \sum_{n \in \mathbb{N}} [x \in D_n] \cdot A(f_n(x), x)\, \mathrm{d}\mu(x) = \int_X \sum_{y \sim x} A(y,x)\, \mathrm{d}\mu(x),$$

as desired. (Here for a statement $\mathfrak{S}$ that can be true or false, the bracketed notation $[\mathfrak{S}]$ stands for 1 if $\mathfrak{S}$ is true and 0 otherwise.)

Exercise 2.5. Verify the above chain of equalities. ∎

Exercise 2.6. Let $\varphi \colon (\ell_2 E)^n \to (\ell_2 E)^n$ be a fibered map. Show that φ can be written in **matrix form**; i.e., there exist unique fibered maps $\varphi^{ij} \colon \ell_2 E \to \ell_2 E$, $1 \leqslant i, j \leqslant n$, such that for all $\mathcal{O} \in X/E$ and $f_1, \ldots, f_n \in \ell_2\mathcal{O}$, we have
$$\varphi_{\mathcal{O}}(f_1, \ldots, f_n) = \left(\sum_{j=1}^n \varphi_{\mathcal{O}}^{1j}(f_j), \ldots, \sum_{j=1}^n \varphi_{\mathcal{O}}^{nj}(f_j)\right).$$

Definition 2.7. Let $\varphi \colon (\ell_2 E)^n \to (\ell_2 E)^n$ be a fibered map and let $\varphi^{ij} \colon \ell_2 E \to \ell_2 E$, $1 \leqslant i, j \leqslant n$ be the fibered maps given by Exercise 2.6 applied to φ. Define the **trace** $\mathrm{tr}(\varphi)$ of φ by the formula
$$\mathrm{tr}(\varphi) := \sum_{i=1}^n \mathrm{tr}(\varphi^{ii}).$$

Lemma 2.8. *Let φ, $\psi\colon (\ell_2 E)^n \to (\ell_2 E)^n$ be fibered maps. Then*

$$\mathrm{tr}(\varphi \circ \psi) \;=\; \mathrm{tr}(\psi \circ \varphi).$$

Exercise 2.9. Prove Lemma 2.8.

3. Von Neumann dimension

Exercise 3.1. Let M be a Hilbert E-module and let $N \subseteq M$ be a submodule of M. For each $\mathcal{O} \in X/E$, let $\mathrm{proj}_{\mathcal{O}}^N\colon M_{\mathcal{O}} \to N_{\mathcal{O}}$ denote the orthogonal projection of $M_{\mathcal{O}}$ onto $N_{\mathcal{O}}$. Show that this defines a fibered map $\mathrm{proj}^N\colon M \to N$, which we call the **projection** of M onto N.

Definition 3.2. Let $M \subseteq (\ell_2 E)^n$ be a Hilbert E-module and let $\mathrm{proj}^M\colon (\ell_2 E)^n \to M$ be the corresponding projection map. Define the **von Neumann dimension** $\dim_E M$ of M by the formula

$$\dim_E M \;\coloneqq\; \mathrm{tr}(\mathrm{proj}^M),$$

where proj^M is viewed as a fibered map $(\ell_2 E)^n \to (\ell_2 E)^n$.

Exercise 3.3. Let M be a Hilbert E-module. Show that $\dim_E M \geqslant 0$.

Exercise 3.4. Show that $\dim_E(\ell_2 E)^n = n$.

Exercise 3.5. Let M be a Hilbert E-module and let $N \subseteq M$ be a submodule of M. Show that $\dim_E N \leqslant \dim_E M$.

Exercise 3.6. Let M be a Hilbert E-module. Show that if every E-class is finite, then

$$\dim_E M \;=\; \int_X \frac{\dim_{\mathbb{R}} M_{[x]}}{|[x]|} \, \mathrm{d}\mu(x).$$

Exercise 3.7. Let $M \subseteq (\ell_2 E)^n$ be a Hilbert E-module. If $m \geqslant n$, then we can embed $(\ell_2 E)^n$ into $(\ell_2 E)^m$ in the natural way. Let M' be the image of M under this embedding (so M' is a submodule of $(\ell_2 E)^m$). Show that $\dim_E M = \dim_E M'$.

Theorem 3.8. *Von Neumann dimension is an isomorphism invariant; i.e., if $M \subseteq (\ell_2 E)^m$ and $N \subseteq (\ell_2 E)^n$ are Hilbert E-modules and $\varphi\colon M \to N$ is an isomorphism, then $\dim_E M = \dim_E N$.*

Proof. After embedding $(\ell_2 E)^n$ and $(\ell_2 E)^m$ into $(\ell_2 E)^n \oplus (\ell_2 E)^m = (\ell_2 E)^{n+m}$ in the obvious way, we may assume that $n = m$ (see Exercise 3.7).

Extend φ to a fibered map $\overline{\varphi}\colon (\ell_2 E)^n \to (\ell_2 E)^n$ by putting

$$\overline{\varphi}|_M \coloneqq \varphi \qquad \text{and} \qquad \overline{\varphi}|_{M^\perp} \coloneqq 0,$$

where $M^\perp$ denotes the (fiber-wise) orthogonal complement of M in $(\ell_2 E)^n$. Let $\overline{\varphi}^*\colon (\ell_2 E)^n \to (\ell_2 E)^n$ denote the (fiber-wise) adjoint of $\overline{\varphi}$. Then

$$\overline{\varphi}^* \circ \overline{\varphi} = \mathrm{proj}^M, \qquad \text{while} \qquad \overline{\varphi} \circ \overline{\varphi}^* = \mathrm{proj}^N,$$

and we are done by Lemma 2.8. ∎

4. Proof of Theorem 0.2

Let $f_1, \ldots, f_n$ be a finite collection of Borel partial involutions $X \rightharpoonup X$ such that:

- $f_i(x) \neq x$ for all i and $x \in \mathrm{dom}(f_i)$; and
- $f_i(x) \neq f_j(x)$ for all $i \neq j$ and $x \in \mathrm{dom}(f_i) \cap \mathrm{dom}(f_j)$.

We say that $f_1, \ldots, f_n$ is a **non-redundant family**. We use $\mathbb{G}(f_1, \ldots, f_n)$ to denote the graph **generated** by $f_1, \ldots, f_n$:

$$x \, \mathbb{G}(f_1, \ldots, f_n) \, y \; :\Longleftrightarrow \; x \in \mathrm{dom}(f_n) \text{ and } y = f_n(x) \text{ for some } n.$$

Exercise 4.1. If G is a Borel graph on X and $\Delta(G)$ is finite, then G is generated by a finite non-redundant family.

Suppose that $G := \mathbb{G}(f_1, \ldots, f_n)$ is a graphing of E. The corresponding Hilbert E-module $\mathbb{M}(f_1, \ldots, f_n) \subseteq (\ell_2 E)^n$ is defined as follows: For each $\mathcal{O} \in X/E$, the space $\mathbb{M}_{\mathcal{O}}(f_1, \ldots, f_n)$ consists of all tuples $(a_1, \ldots, a_n) \in (\ell_2 E)^n$ such that for every $x \in \mathcal{O}$,

    – $x \notin \mathrm{dom}(f_i)$ implies $a_i(x) = 0$; and

    – $x \in \mathrm{dom}(f_i)$ implies $a_i(x) = -a_i(f_i(x))$.

The elements of $\mathbb{M}_{\mathcal{O}}(f_1, \ldots, f_n)$ can be thought of as ℓ_2-combinations of the edges in $G|_{\mathcal{O}}$, where an edge (x, y) is interpreted as $-(y, x)$.

Lemma 4.2. *Let f_1, $\ldots$, f_n be a non-redundant family such that $G := \mathbb{G}(f_1, \ldots, f_n)$ is a graphing of E and let $M := \mathbb{M}(f_1, \ldots, f_n)$. Then*

$$C_\mu(G) = \dim_E M.$$

Proof. Let $\pi^{ij} \colon \ell_2 E \to \ell_2 E$ be the matrix coefficients of the projection map $\mathrm{proj}^M \colon (\ell_2 E)^n \to M$. For any $x \in X$, we have

$$\pi^{ii}_{[x]}(1_x) = \begin{cases} (1/2) \cdot 1_x - (1/2) \cdot 1_{f_i(x)} & \text{if } x \in \mathrm{dom}(f_i); \\ 0 & \text{if } x \notin \mathrm{dom}(f_i). \end{cases}$$

Exercise 4.3. Verify the above equality.

Since the family f_1, $\ldots$, f_n is non-redundant, we conclude that

$$\sum_{i=1}^n \pi^{ii}_{[x]}(1_x)(x) = \frac{1}{2} \deg_G(x).$$

Thus,

$$\dim_E M = \mathrm{tr}(\mathrm{proj}^M) = \sum_{i=1}^n \mathrm{tr}(\pi^{ii}) = \int_X \sum_{i=1}^n \pi^{ii}_{[x]}(1_x)(x) \, \mathrm{d}\mu(x) = \frac{1}{2} \int_X \deg_G(x) \, \mathrm{d}\mu(x) = C_\mu(G),$$

as desired. ■

Now we are ready to prove Theorem 0.2.

Proof of Theorem 0.2. Let T and G be as in the statement of Theorem 0.2. Since $\Delta(T)$ and $\Delta(G)$ are finite, there are finite non-redundant families of partial Borel involutions f_1, $\ldots$, f_m and g_1, $\ldots$, g_n such that $G = \mathbb{G}(f_1, \ldots, f_m)$ and $T = \mathbb{G}(g_1, \ldots, g_n)$. Let $M := \mathbb{M}(f_1, \ldots, f_m)$ and $N := \mathbb{M}(g_1, \ldots, g_n)$. In view of Lemma 4.2, we wish to show that $\dim_E N \leqslant \dim_E M$.

A *G-representation* of an edge $(x, y) \in T$ is an xy-path in G. Note that every edge of T admits a G-representation of length at most c, where c is the Lipschitz constant from the statement of Theorem 0.2. Since there are only finitely many xy-paths in G of length at most c, we can choose one such G-representation $P(x, y)$ in a Borel fashion. Furthermore, we may assume that $P(y, x)$ is the reverse of $P(x, y)$.

Now let $\mathcal{O} \in X/E$ and consider the spaces $N_{\mathcal{O}}$ and $M_{\mathcal{O}}$. Their elements can be naturally identified with the ℓ_2-combinations of the edges in $T|_{\mathcal{O}}$ and $G|_{\mathcal{O}}$ respectively, and we can use this identification to define a map $\varphi_{\mathcal{O}} \colon N_{\mathcal{O}} \to M_{\mathcal{O}}$ by sending the characteristic function of every edge $(x, y) \in T|_{\mathcal{O}}$ to the sum of the characteristic functions of the edges on $P(x, y)$, and extending to all of $N_{\mathcal{O}}$ by linearity. Note that $\|\varphi_{\mathcal{O}}\| \leqslant \sqrt{c}$, so this gives a fibered map $\varphi \colon N \to M$.

The key observation is that, since T is acyclic, $\varphi_{\mathcal{O}}$ is injective for every $\mathcal{O} \in X/E$.

Exercise 4.4. Prove this claim. (HINT: Each edge of G has a unique T-representation.)

Therefore, by Theorem 1.5, N is isomorphic to the submodule of M that associates to each $\mathcal{O} \in X/E$ the closure of $\mathrm{im}(\varphi_{\mathcal{O}})$. Hence, $\dim_E N \leqslant \dim_E M$ by Exercise 3.5, and the proof is complete. ■

References

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